Casimir torque

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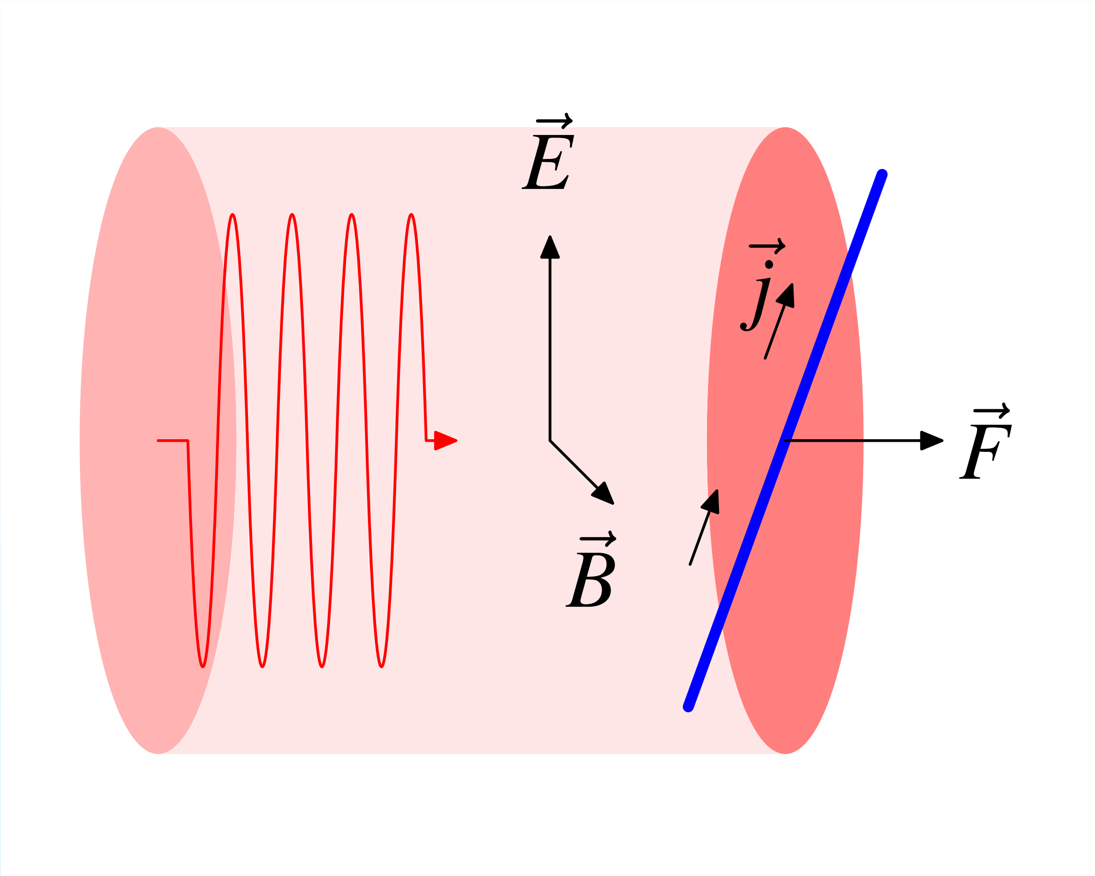
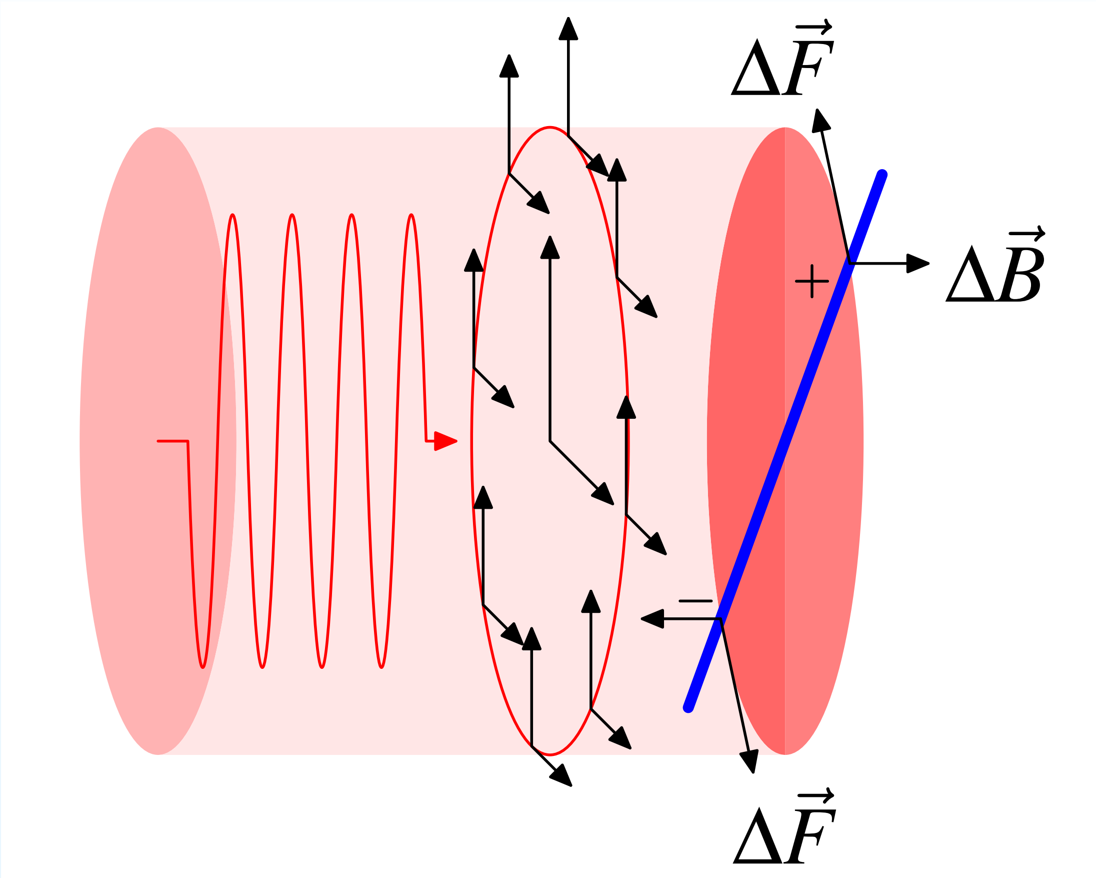
Electromagnetic torque



$$\vec{j} \times \vec{B} \longrightarrow \vec{F}$$

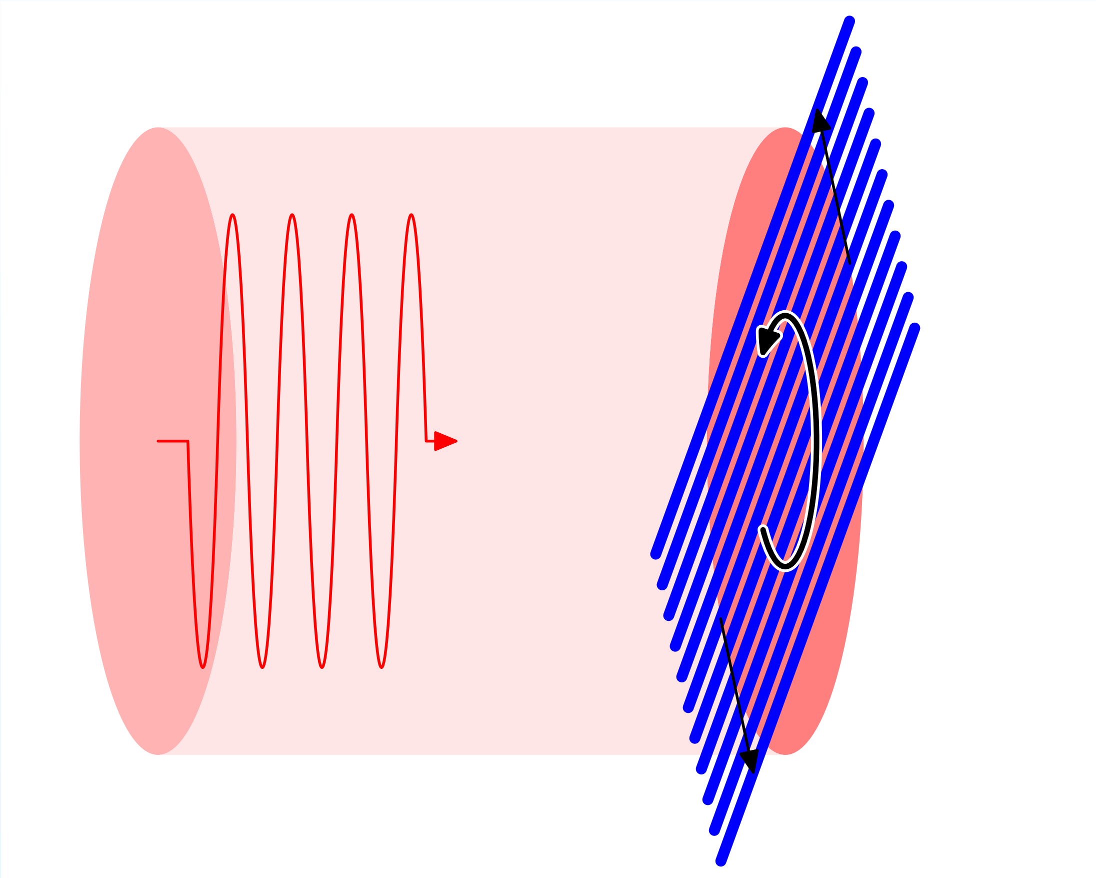
Electromagnetic torque



$\nabla \cdot \vec{j} \neq 0$, $\nabla_\| \cdot \vec{B}_\| \neq 0$.

$\rho \neq 0$ close to edge; $\Delta\vec{B}$ along $\vec{k}$ so that $\nabla \cdot \vec{B} = 0$.

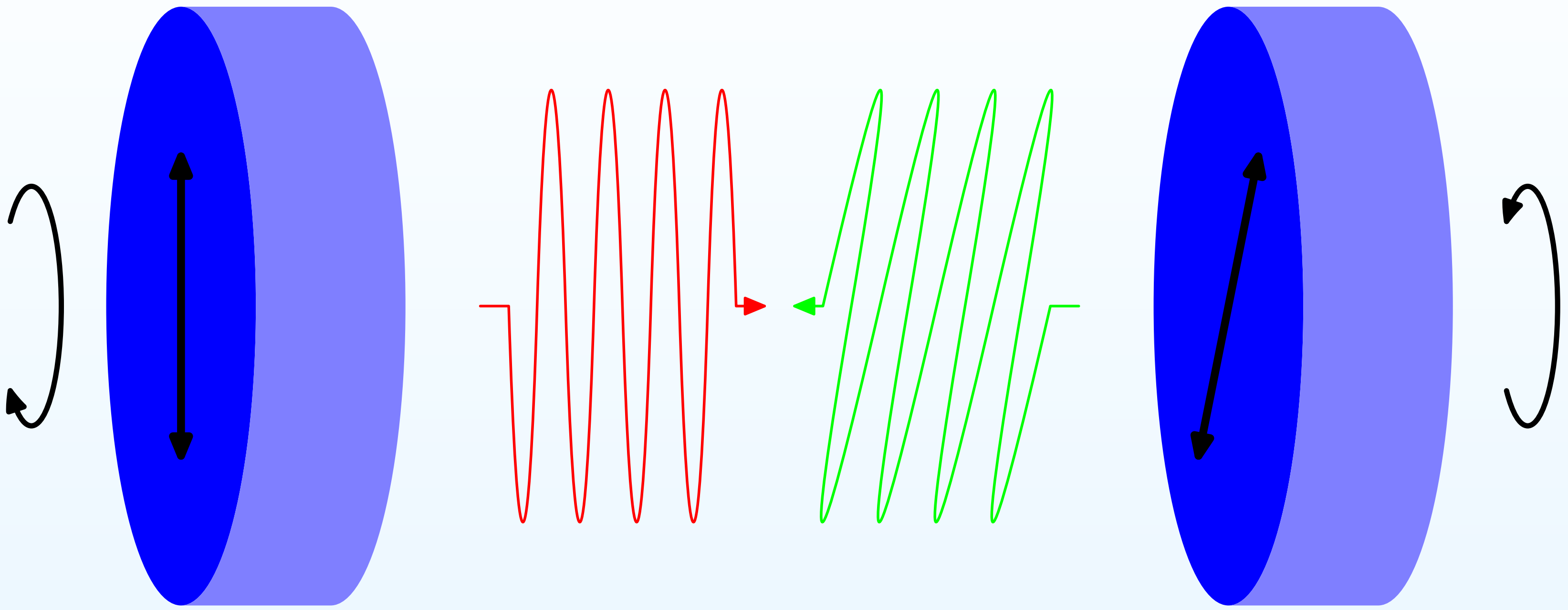
Electromagnetic torque



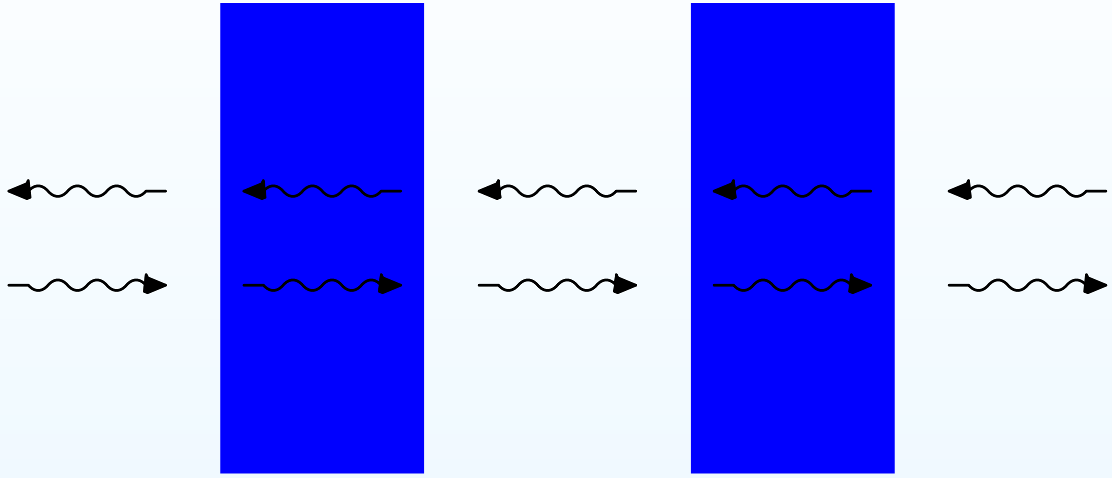
$\vec{j} \times \Delta\vec{B}$ produces a torque on anisotropic media. $\Delta B, \Delta F \propto 1/R$,

$\tau \propto R\Delta F$ independent of width.

Torque in a cavity



Energy



- $\nabla^2 \vec{A} + \epsilon_a(\omega)\frac{\omega^2}{c^2}\vec{A} = 0$ + B.C. $\Rightarrow$ normal modes ω_k.

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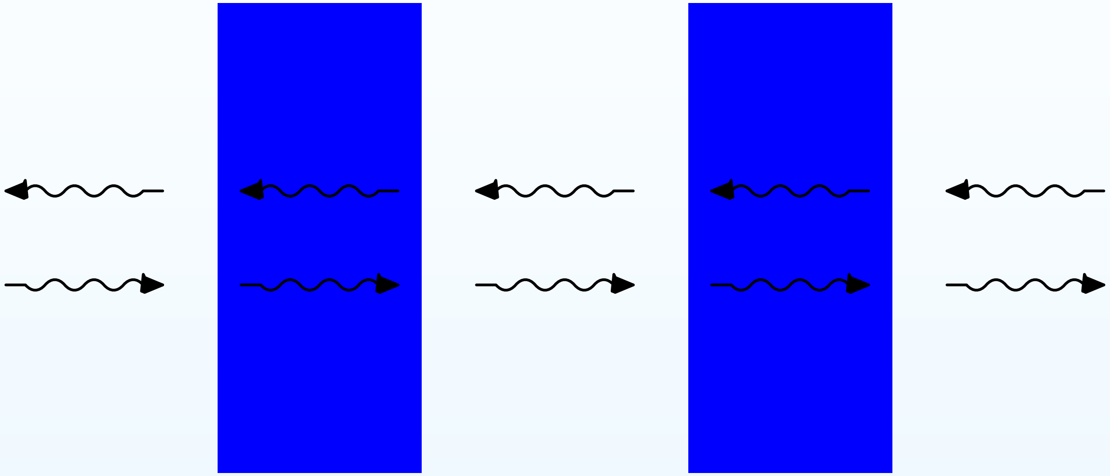

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- $U = \sum(\overline{n}_k + 1/2)\hbar\omega_k$

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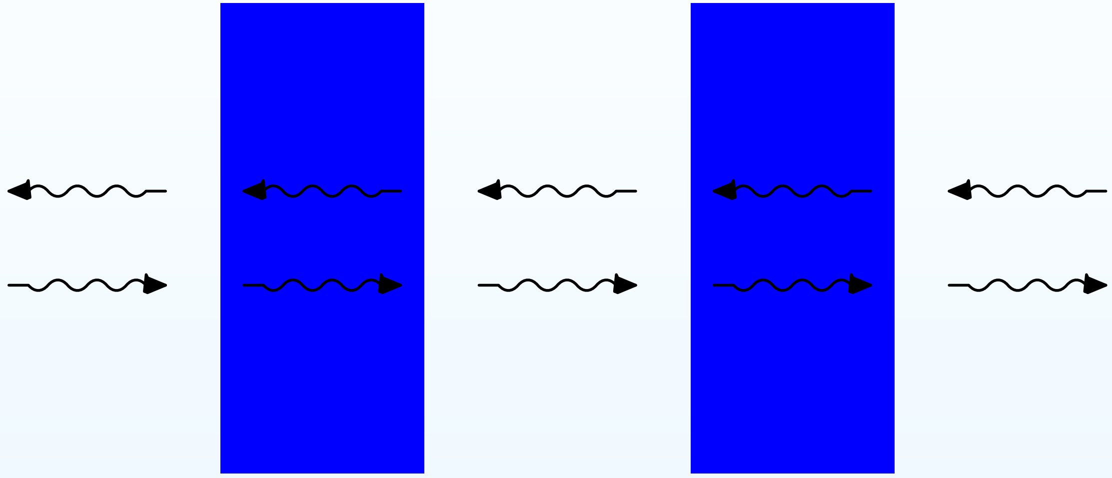

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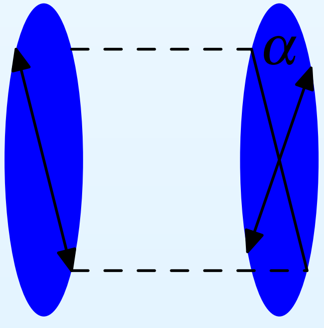
- $U = \sum(\overline{n}_k + 1/2)\hbar\omega_k$
- $\tau = -\dfrac{\partial}{\partial\alpha}U$



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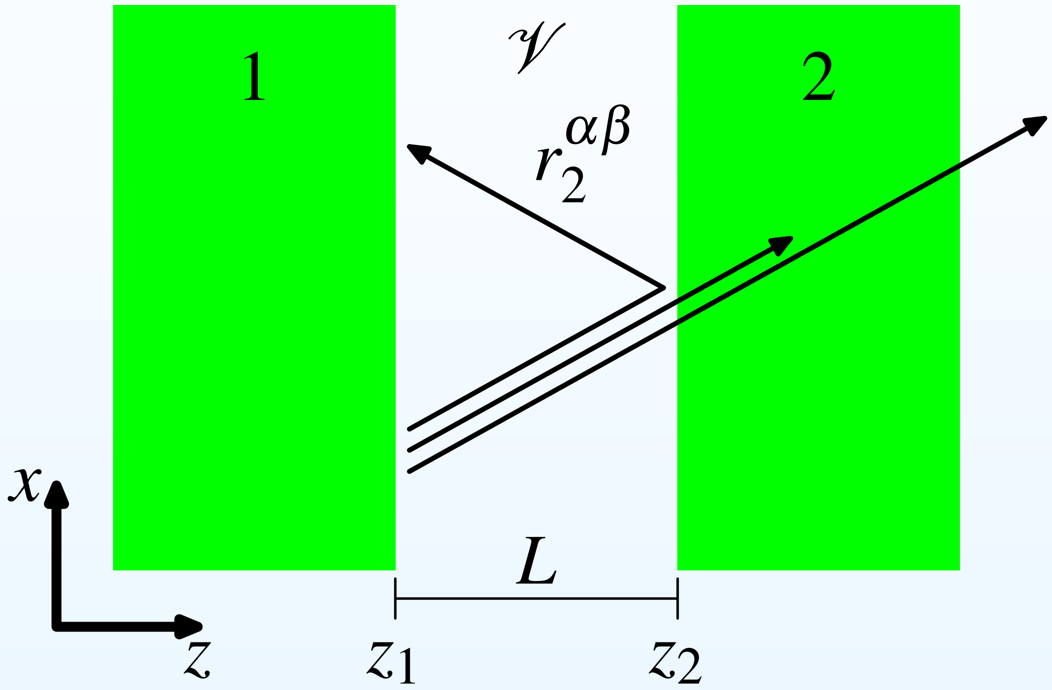
- Usual assumptions: local, homogeneous, isotropic system, sharp boundaries…

Alternative setup

- Real system:



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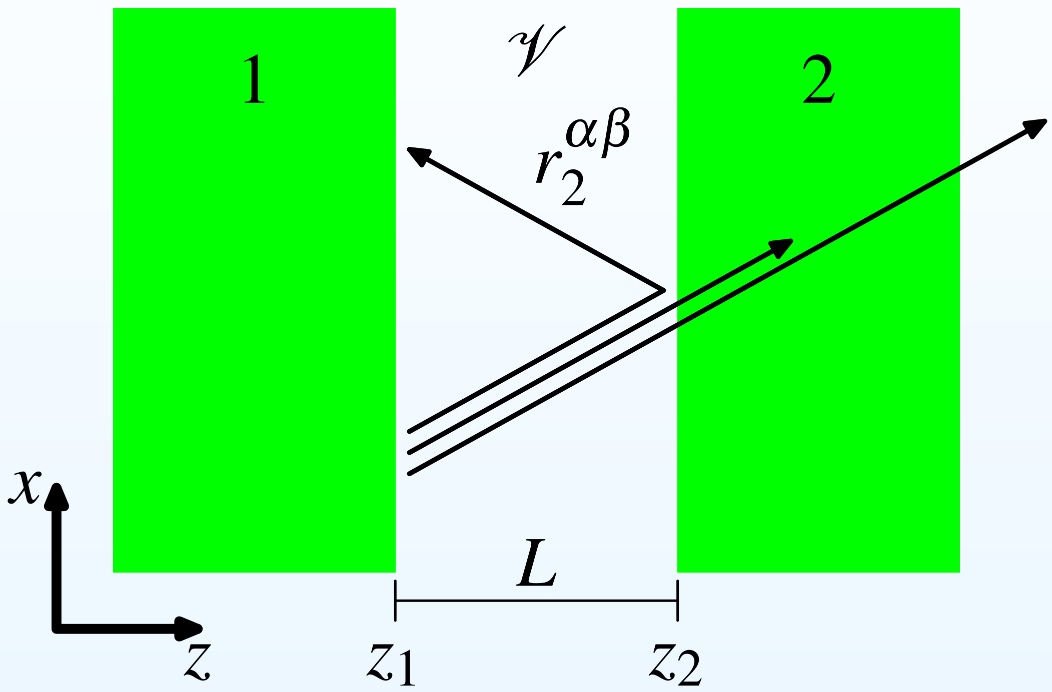

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- Exact surface impedance tensor $\hat{n} \times (\hat{n} \times \vec{E}_2) = \mathbf{Z}_2 \cdot \hat{n} \times \vec{H}_2$

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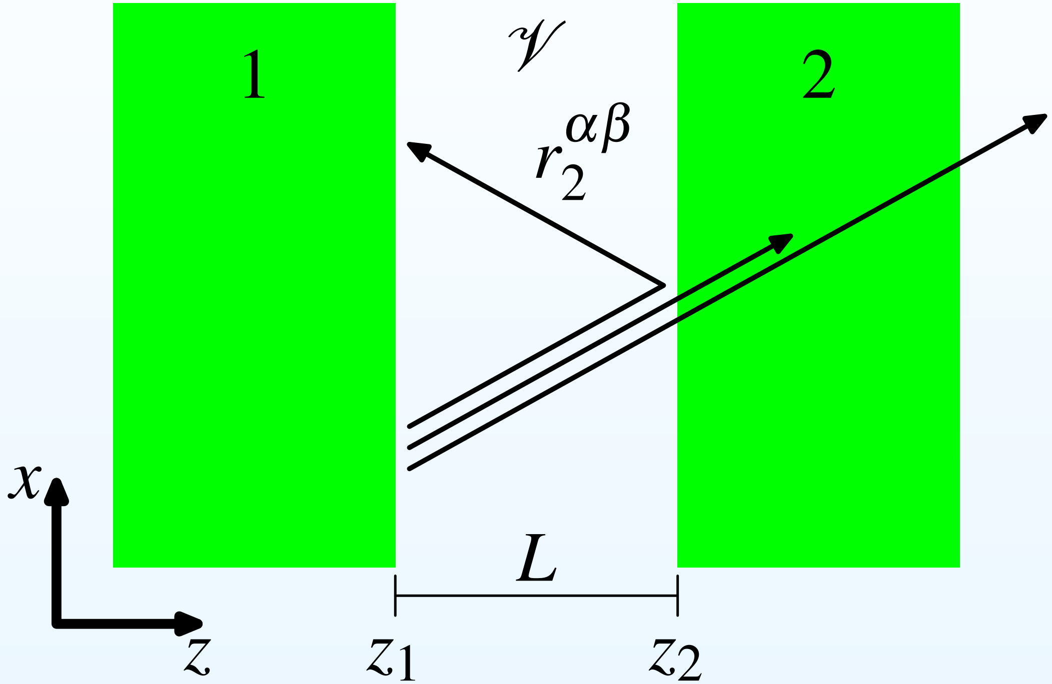

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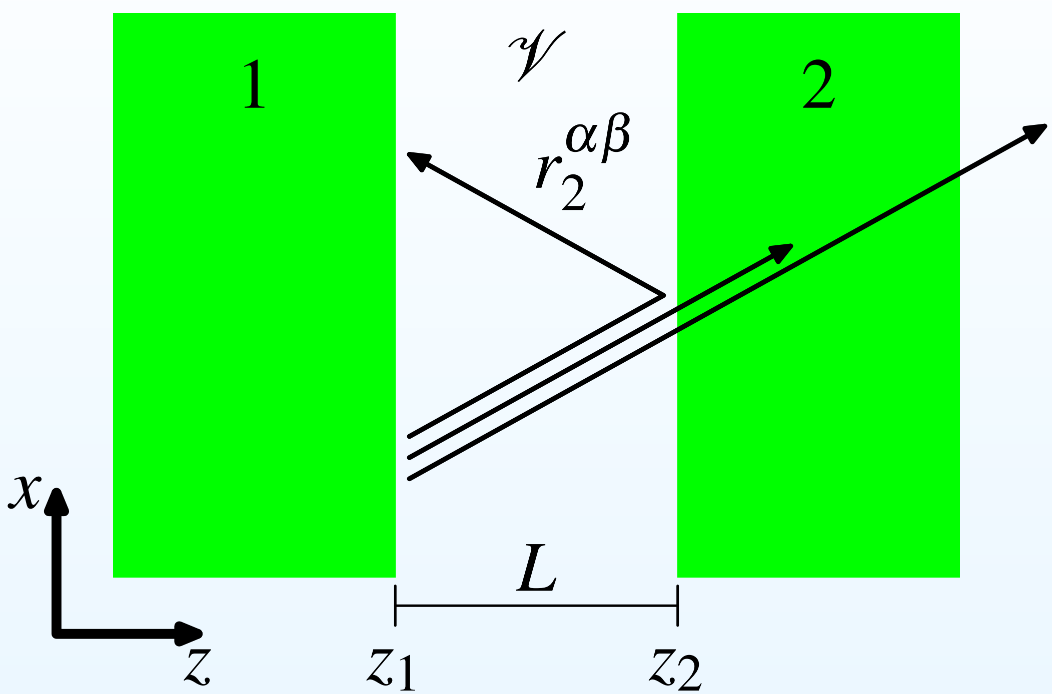

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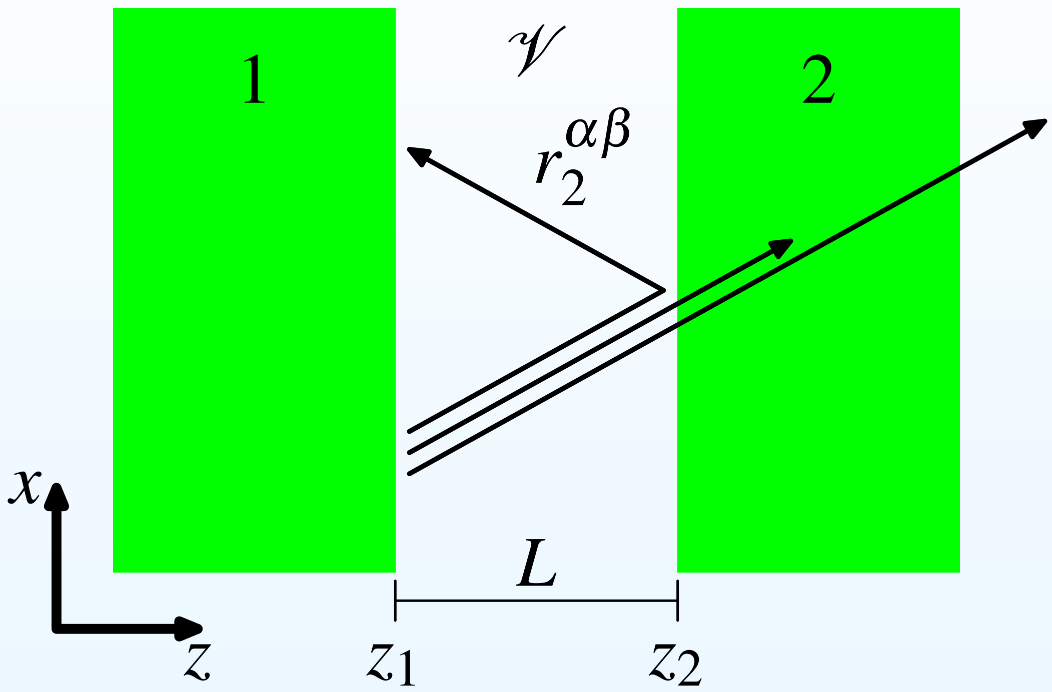

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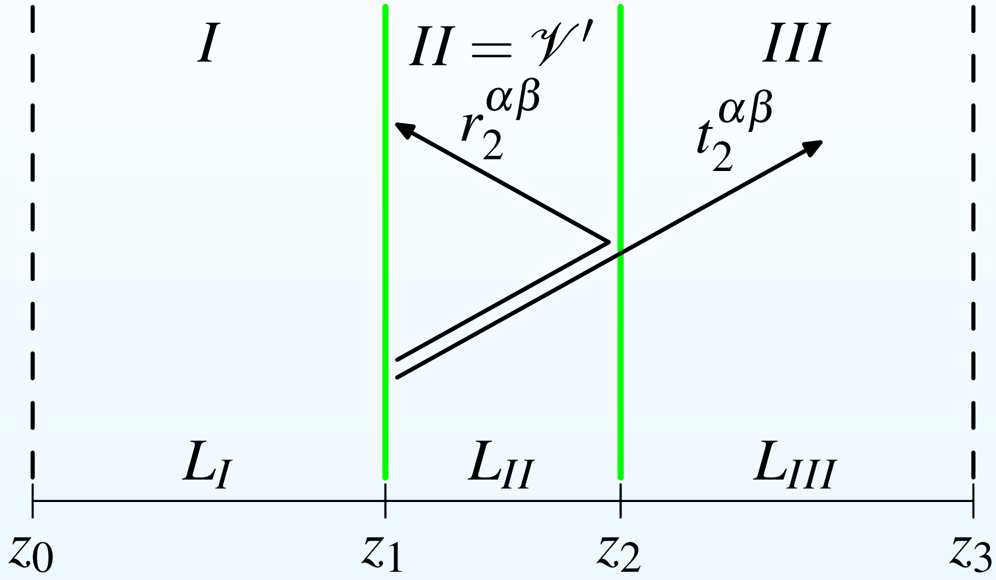
- $\Rightarrow$ Coherent reflection amplitude $r_2^{\alpha\beta}$.
- Detailed balance: Incoherent emission $\propto 1 - |r_2|^2$.
- Within the cavity, *everything depends exclusively on* $\mathbf{Z}_a$, $\mathbf{r}_a$*!*

Fictitious system

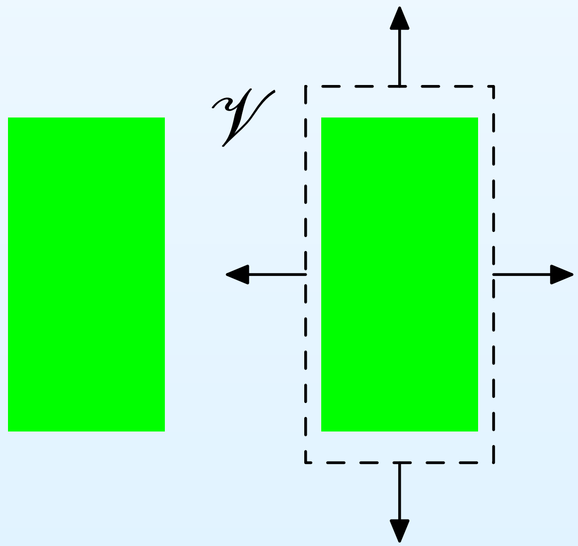


- Chose $r_a^{\alpha\beta}$ identical to those of the real system,
- $t_a^{\alpha\beta}$ such that energy is conserved: no absorption, no degrees of freedom beyond e.m.!
- Appropriate BC at z_0, z_3 to quantize and count normal modes. . . .
- $L_{II} \ll L_I, L_{III} \to \infty$.

Mechanical properties of the e.m. field

Identical within fictitious cavity $\mathcal{V}'$ to those within the real cavity $\mathcal{V}$.

- Energy density: $u = (E^2 + B^2)/8\pi$
- Energy flux: $\vec{S} = \frac{c}{4\pi}\vec{E} \times B$
- Momemtum flux: $-\mathbf{T} = \frac{1}{4\pi}\left[\vec{E}\vec{E} + \vec{B}\vec{B} - \frac{1}{2}(E^2 + B^2)\mathbf{1}\right]$
- Angular momentum flux: $-\mathbf{M} = -\vec{r} \times \mathbf{T}$
- $\vec{\tau} = \int \mathbf{M} \cdot d\vec{a}$



Advantages

⇒ expressions in terms of exact surface impedances or, equivalently, reflection amplitudes. Dissipationless, homogeneous, isotropic, local, sharp media may be treated on the same footing as dissipative, inhomogeneous (layered, superlattice, photonic structures,. . .), anisotropic, chiral, spatially dispersive, smooth media.

Wide finite beam: 1D

$$\vec{E} = E_{>}\hat{e}_{>}e^{i(qz-\omega t)} + E_{<}\hat{e}_{<}e^{-i(qz+\omega t)}$$

$$\vec{B} = B_{>}\hat{b}_{>}e^{i(qz-\omega t)} + B_{<}\hat{b}_{<}e^{-i(qz+\omega t)}$$

Wide finite beam: 1D

$$\vec{E} = E_> \hat{e}_> e^{i(qz-\omega t)} + E_< \hat{e}_< e^{-i(qz+\omega t)} - \frac{\hat{e}_> \cdot \nabla E_>}{iq} \hat{z} e^{i(qz-\omega t)} + \frac{\hat{e}_< \cdot \nabla E_<}{iq} \hat{z} e^{-i(qz+\omega t)}$$

$$\vec{B} = B_> \hat{b}_> e^{i(qz-\omega t)} + B_< \hat{b}_< e^{-i(qz+\omega t)} - \frac{\hat{b}_> \cdot \nabla B_>}{iq} \hat{z} e^{i(qz-\omega t)} + \frac{\hat{b}_< \cdot \nabla B_<}{iq} \hat{z} e^{-i(qz+\omega t)}$$

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$$\tau_z = \frac{1}{8\pi} \mathrm{Re} \int da [(\vec{r} \times \vec{E}^*)_z E_z + (\vec{r} \times \vec{B}^*)_z B_z]$$

after some manipulation. . .

$$\tau_z = \frac{1}{8\pi q}\text{Im}\int da\vec{E}\cdot\vec{B}^* \to \frac{A}{8\pi q}\text{Im}\vec{E}\cdot\vec{B}^* = -\frac{1}{8\pi q^2}\text{Re}(\vec{E}\times\frac{\partial}{\partial z}\vec{E}^*)\cdot\hat{z}$$

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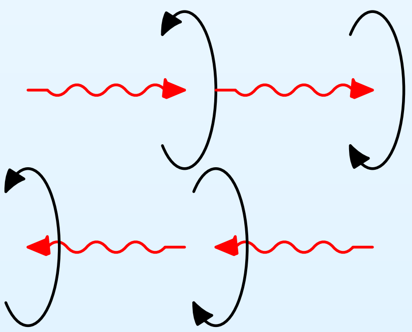
Interpretation:

$$\begin{aligned}\vec{E} &= (E_{>+}\hat{e}_+ + E_{>-}\hat{e}_-)e^{i(qz-\omega t)}\\ &\quad +(E_{<+}\hat{e}_+ + E_{<-}\hat{e}_-)e^{-i(qz+\omega t)}\end{aligned}$$

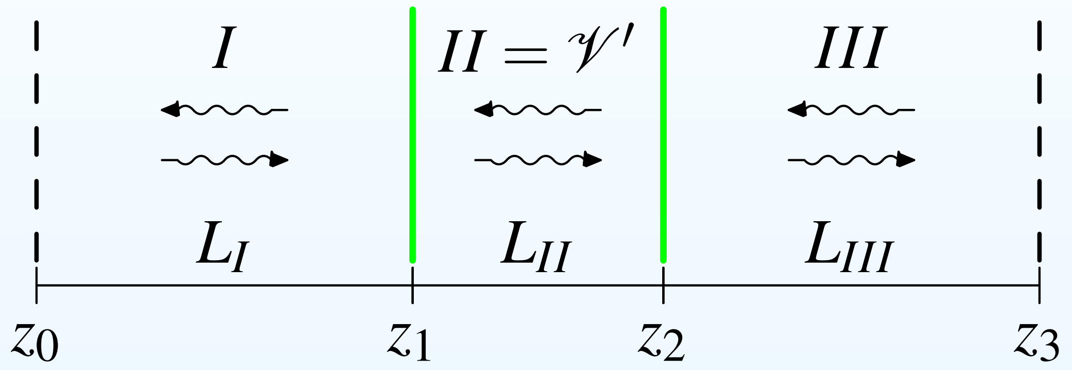
$$\frac{\tau_z}{A} = c\,\frac{1}{\omega}\,\frac{1}{8\pi}\left(|E_{>+}|^2 - |E_{>-}|^2 - |E_{<+}|^2 + |E_{<-}|^2\right),$$

consistent with $E=\hbar\omega$, $L_z=\pm\hbar$, speed=$\pm c$.



One mode

- $\vec{E} = \mathcal{E}_0 \vec{\phi}(z) e^{-i\omega t}$
- $\vec{\phi}(z) = (\alpha^r_\lambda e^{iqz} + \beta^r_\lambda e^{-iqz})\hat{e}_\lambda$ within region $r =$I, II, III.



- Energy $U = \frac{A}{8\pi}[L_I(|\alpha^I|^2 + |\beta^I|^2) + L_{III}(|\alpha^{III}|^2 + \beta^{III}|^2)]|\mathcal{E}_0|^2$
- and normalization $1 = \frac{A}{8\pi}[L_I(|\alpha^I|^2 + |\beta^I|^2) + L_{III}(|\alpha^{III}|^2 + \beta^{III}|^2)]$
- dominated by large fictitious regions I and III,
- $U = \frac{A}{8\pi}|\mathcal{E}_0|^2$, yielding $|\mathcal{E}_0|^2 = 8\pi f_\omega \hbar\omega/A$, with $f_\omega = \coth(\beta\hbar\omega/2)/2$ (poor man's 2nd quantization).

Contribution to torque

$$\tau_z = \frac{\hbar c}{2q} f_\omega [\phi_x \partial_z \phi_y^* - \phi_y \partial_z \phi_x^* + (\partial_z \phi_y)\phi_x^* - (\partial_z \phi_x)\phi_y^*].$$

Sum over fictitious modes

$$\tau_z = \frac{\hbar c}{2}\sum_n \frac{f_{w_n}}{q_n} \times[\phi_{nx}\partial_z\phi^*_{ny} - \phi_{ny}\partial_z\phi^*_{nx} + (\partial_z\phi_{ny})\phi^*_{nx} - (\partial_z\phi_{nx})\phi^*_{ny}].$$

$$\tau_z = \hbar c\int dq\, f_{qc}\sum_n \delta(q^2 - q_n^2) \times[\phi_{nx}\partial_z\phi^*_{ny} - \phi_{ny}\partial_z\phi^*_{nx} + (\partial_z\phi_{ny})\phi^*_{nx} - (\partial_z\phi_{nx})\phi^*_{ny}].$$

Introduce Green's function

$$G_{\mu\nu}(q^2; z, z') = \sum_n \phi_{n\mu}(z)\phi_{n\nu}(z')/(q^2 + i\eta - q_n^2)$$

Torque vs. Green's function

$$\tau_z = \frac{\hbar c}{\pi} \int_0^\infty dq\, f_{qc} (\partial_{z'} - \partial_z)[A_{xy}(z, z') - A_{yx}(z, z')]_{z=z'}$$

$A_{\mu\nu}(z, z') = [G_{\mu\nu}(z, z') - G_{\nu\mu}(z', z)]/2i =$ anti-Hermitean part of the Green's function of the fictitious system evaluated within the real cavity.

Calculation of G

$G \Leftarrow$ solution of Helmholtz equation within cavity

$$(\partial_z^2 + q^2 + i\eta)G_{\mu\nu}(z,z') = \delta(z-z')\delta_{\mu\nu},$$

+ singular source + boundary conditions,

$$\begin{aligned}\mathbf{G}(z,z') &= \mathbf{u}(z)[\mathbf{u}'(z') - \mathbf{v}'(z')\mathbf{v}^{-1}(z')\mathbf{u}(z')]^{-1}\theta(z-z')\\ &\quad -\mathbf{v}(z)[\mathbf{v}'(z') - \mathbf{u}'(z')\mathbf{u}^{-1}(z')\mathbf{v}(z')]^{-1}\theta(z'-z),\end{aligned}$$

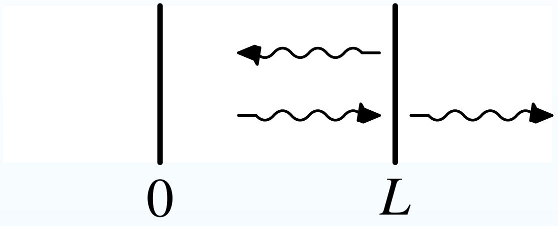
where

$$\mathbf{u} = (\vec{u}_1, \vec{u}_2) = \left(\begin{array}{c|c} u_{x1} & u_{x2} \\ u_{y1} & u_{y2} \end{array}\right), \quad \mathbf{v} = (\vec{v}_1, \vec{v}_2) = \left(\begin{array}{c|c} v_{x1} & v_{x2} \\ v_{y1} & v_{y2} \end{array}\right)$$

are 2+2 homogeneous solutions obeying BC to the right and left.

(Similar to $u(z_>)v(z_<)/W$).

Example: Uniaxial or orthorhombic system



$$\mathbf{u}_0(z) = \left(\begin{array}{c|c} 1 & 0 \\ 0 & 1 \end{array}\right) e^{iq(z-L)} + \left(\begin{array}{c|c} r_{2x} & 0 \\ 0 & r_{2y} \end{array}\right) e^{-iq(z-L)},$$

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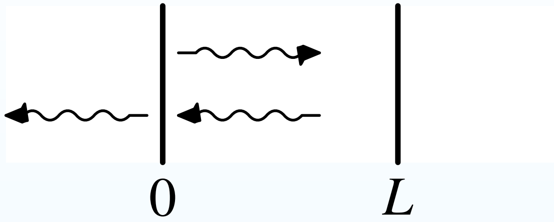

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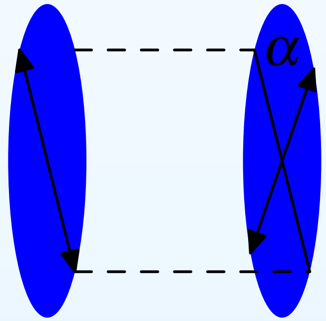


Rotate:

$$\mathbf{u}(z) = \mathbf{R}(\alpha/2)\cdot\mathbf{u}(z), \quad \mathbf{v}(z) = \mathbf{R}(\alpha/2)\cdot\mathbf{v}(z).$$

Substitute, . . .

T=0, 1D

$$\tau_z = -\frac{\hbar c}{2\pi}\int_0^\infty d\kappa \, \frac{\Delta r_1 \Delta r_2 \sin 2\alpha \, e^{-2\kappa L}}{\Delta r_1 \Delta r_2 \sin^2 \alpha e^{-2\kappa L} + (1 - r_{1x} r_{2x} e^{-2\kappa L})(1 - r_{1y} r_{2y} e^{-2\kappa L})},$$

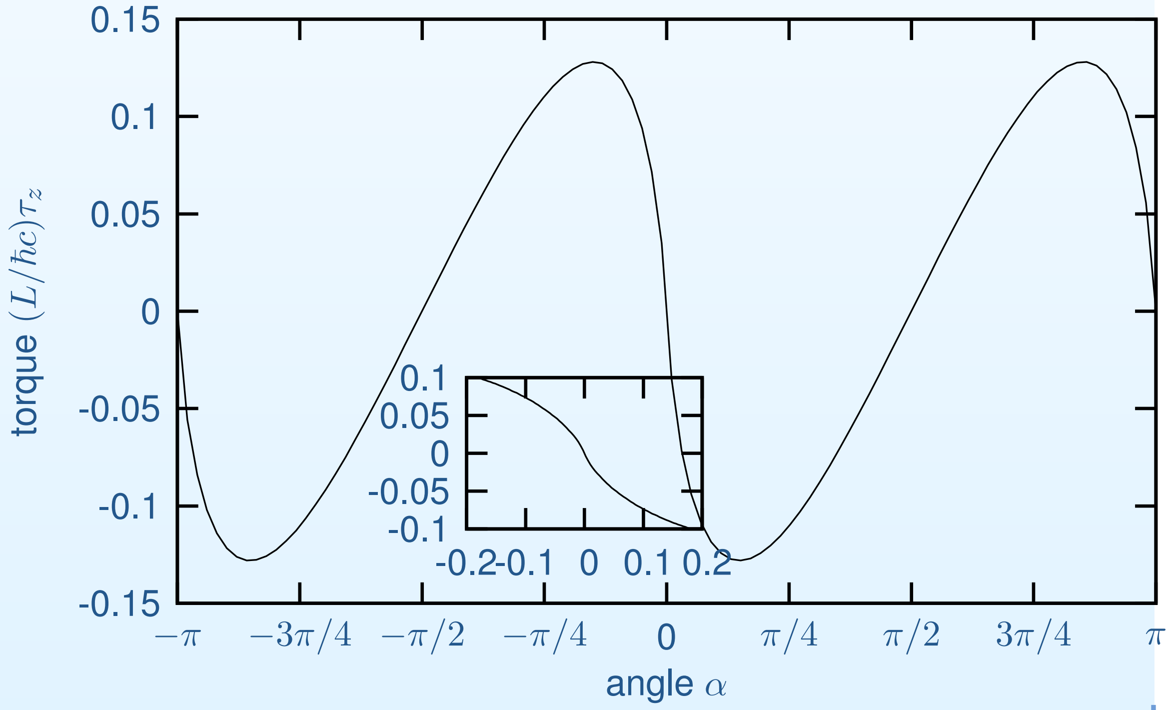
$q = i\kappa$, $\Delta r_a = r_{ax} - r_{ay}$.

Ideal system

Ideal mirror covered by ideal polarizer: $r_{1x} = r_{2x} = 1$, $r_{1y} = r_{2y} = 0$,

$$\tau_z = -\frac{\hbar c \sin 2\alpha}{2\pi} \int_0^\infty \frac{d\kappa}{e^{2\kappa L} - \cos^2(\alpha)} = \frac{\hbar c}{2\pi L} \tan\alpha \log \sin^2 \alpha.$$

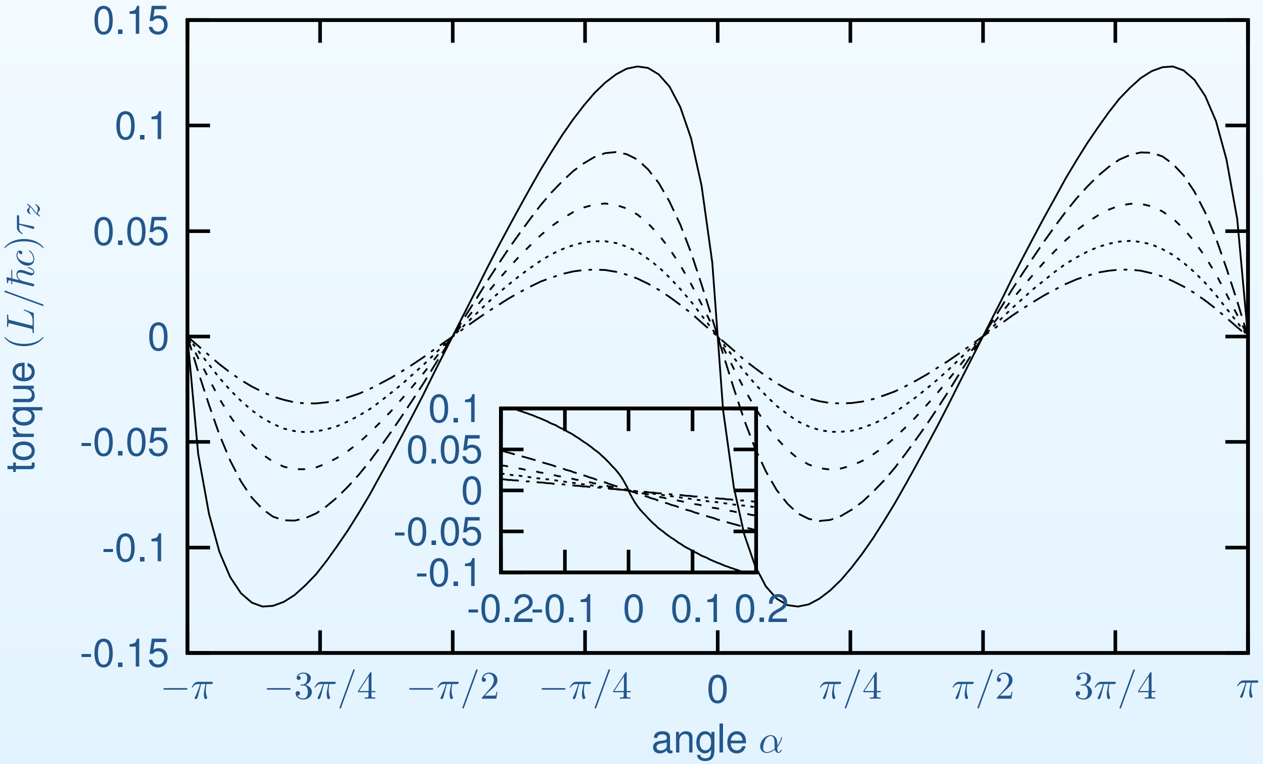
- $\tau \approx 0.1\hbar c/L$
  At 10nm, 3×10^{-19}Nm
- $\tau_{3D} \sim \tau_{1D} A/L^2$
- Asymmetry.
- Singularity.



Lossy mirrors

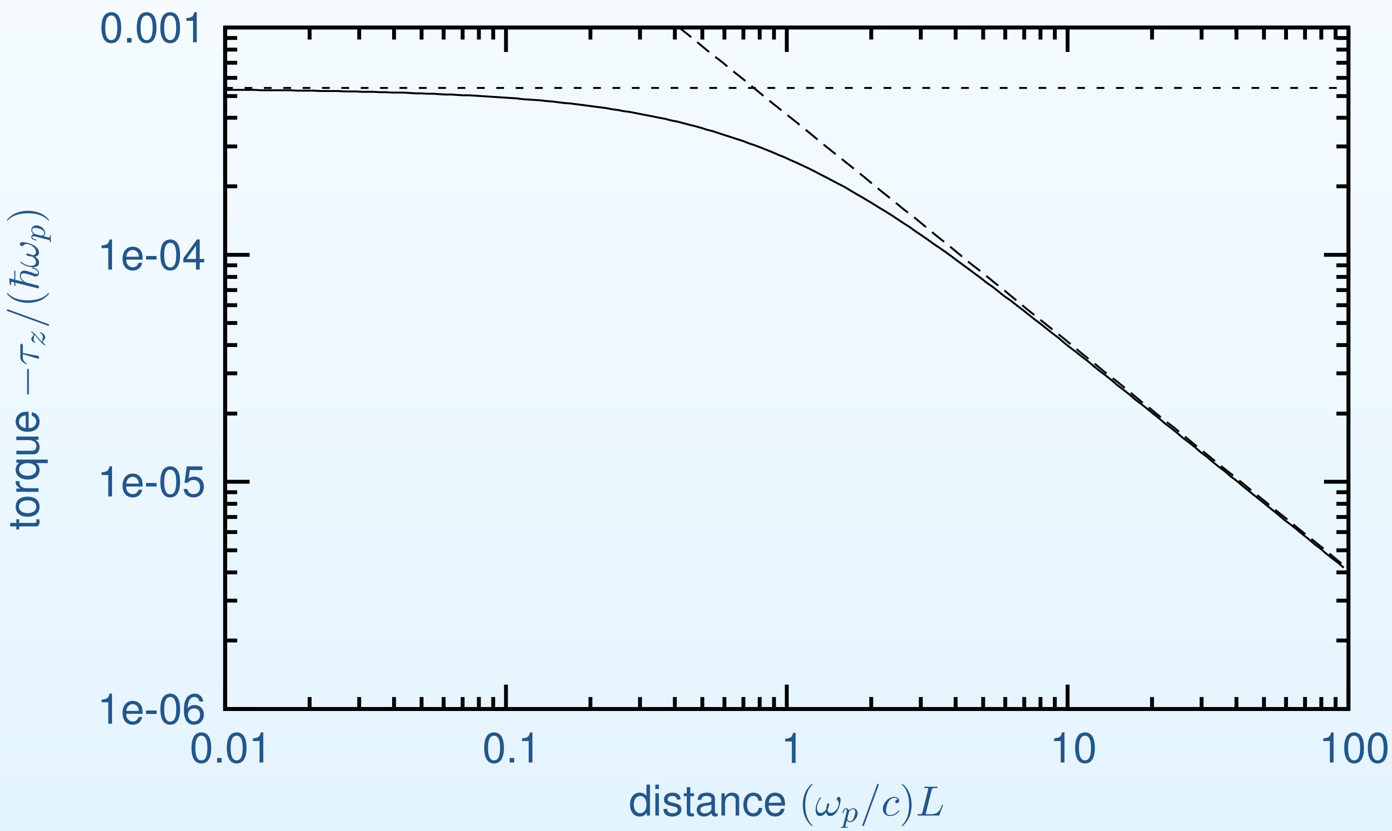
$r_{1x} = r_{2x} = r$ and $r_{1y} = r_{2y} = 0$.

$$\tau_z = \frac{\hbar c r^2 \sin 4\theta}{2\pi} \int_0^\infty d\kappa \, \frac{1}{r^2(\cos^2 2\theta) - e^{2\kappa L}} = \frac{\hbar c \tan\alpha}{2\pi L} \log(1 - r^2 \cos^2 \alpha),$$



Dicroic mirrors

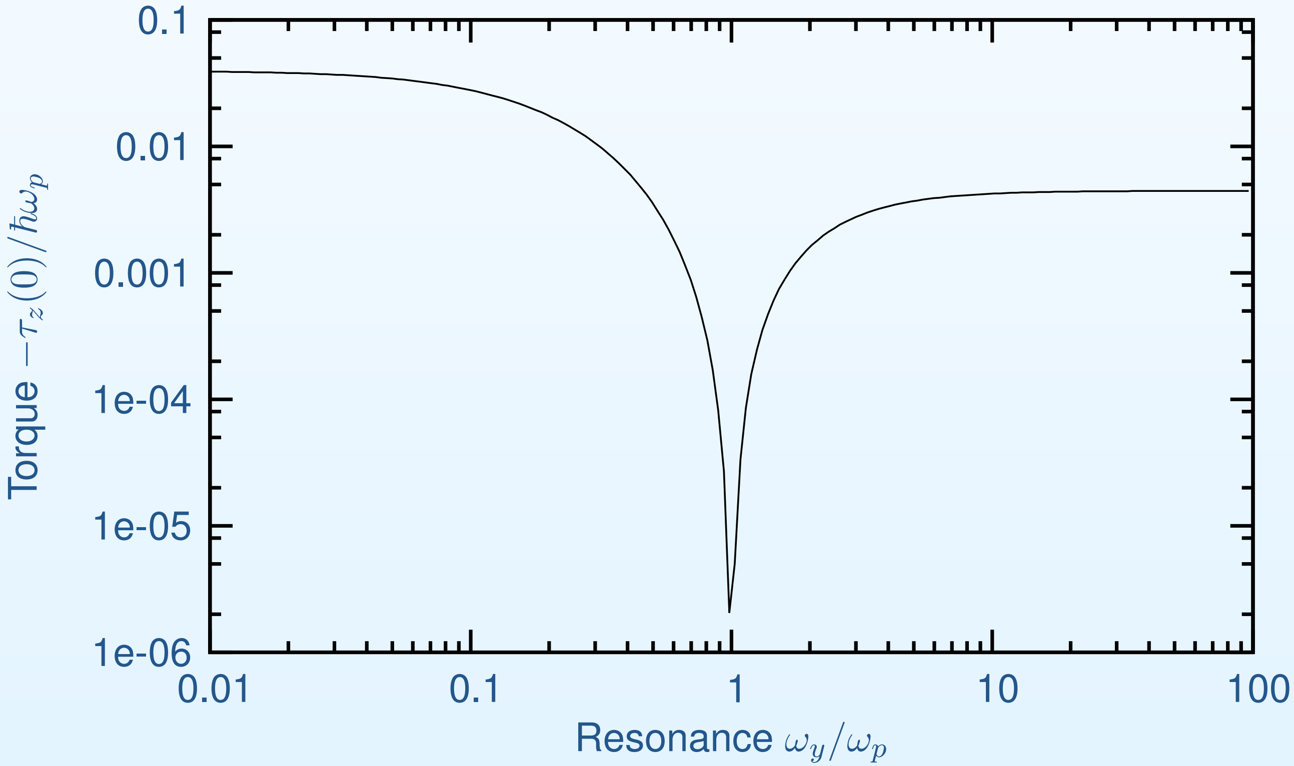
$$\epsilon_i(\omega) = 1 + \frac{\omega_{ip}^2}{\omega_i^2 - \omega^2 - i\omega\tau_i},$$



$\omega_x = \omega_p$, $\omega_y = \sqrt{2}\omega_p$, $\alpha = \pi/4$.

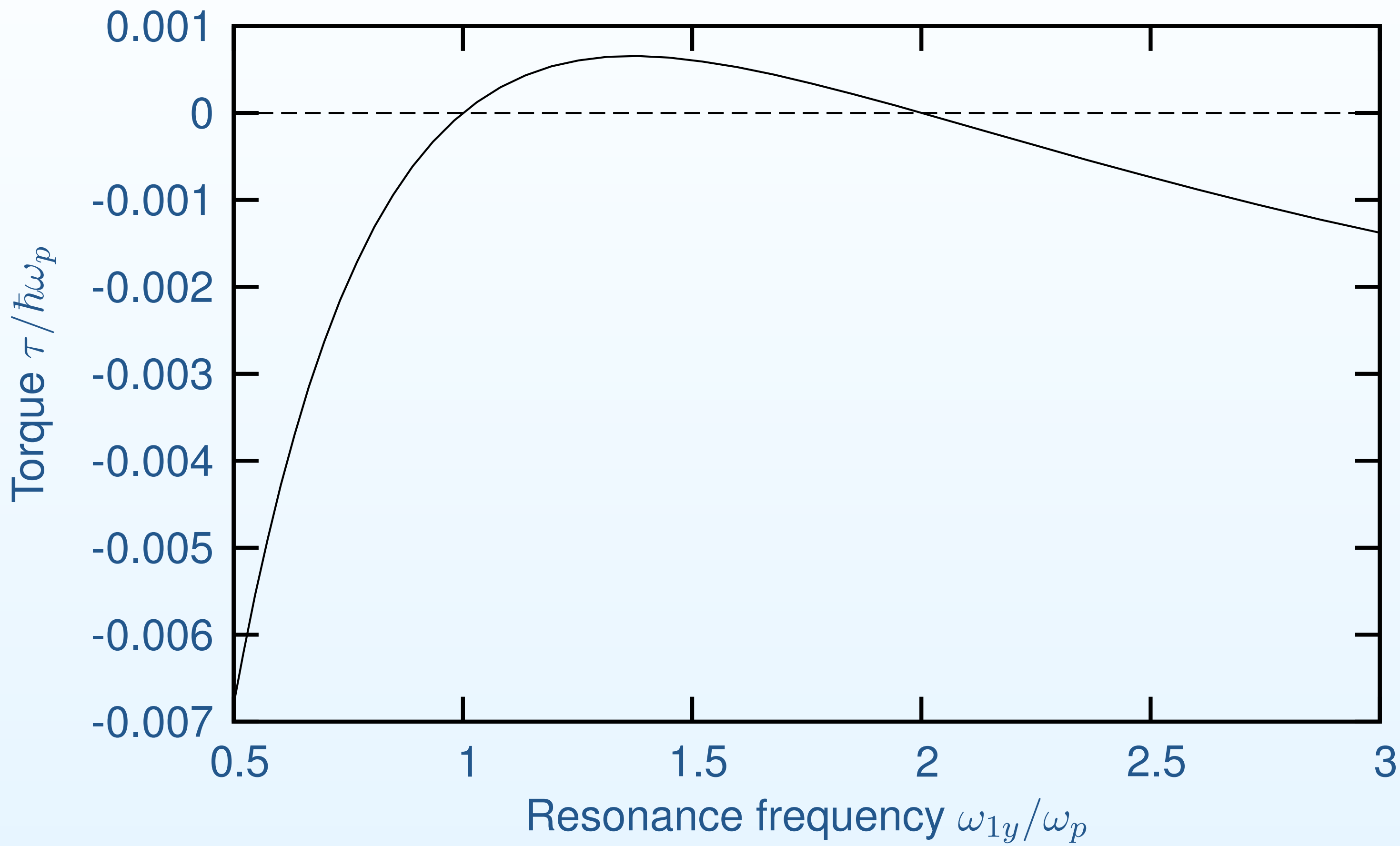
$L = 0$

$$\tau_z(0) = -\frac{\hbar c}{2\pi}\int_0^\infty d\kappa\, \frac{\Delta r^1 \Delta r^2 \sin 2\alpha}{\Delta r^1 \Delta r^2 \sin^2\alpha + (1 - r_x^1 r_x^2)(1 - r_y^1 r_y^2)},$$



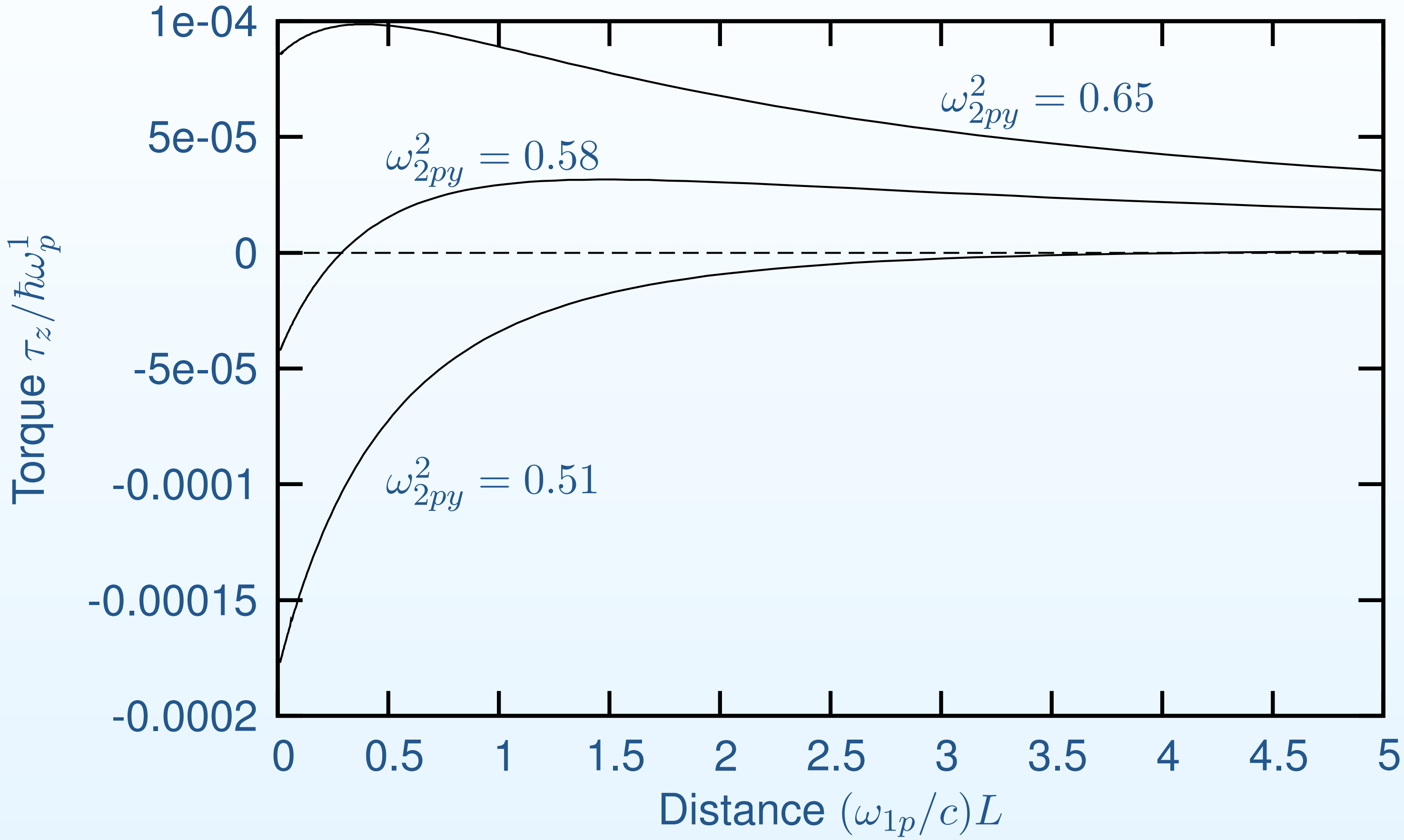
$\omega_x = \omega_p$, $\alpha = \pi/4$, $L = 0$

Dissimilar mirrors



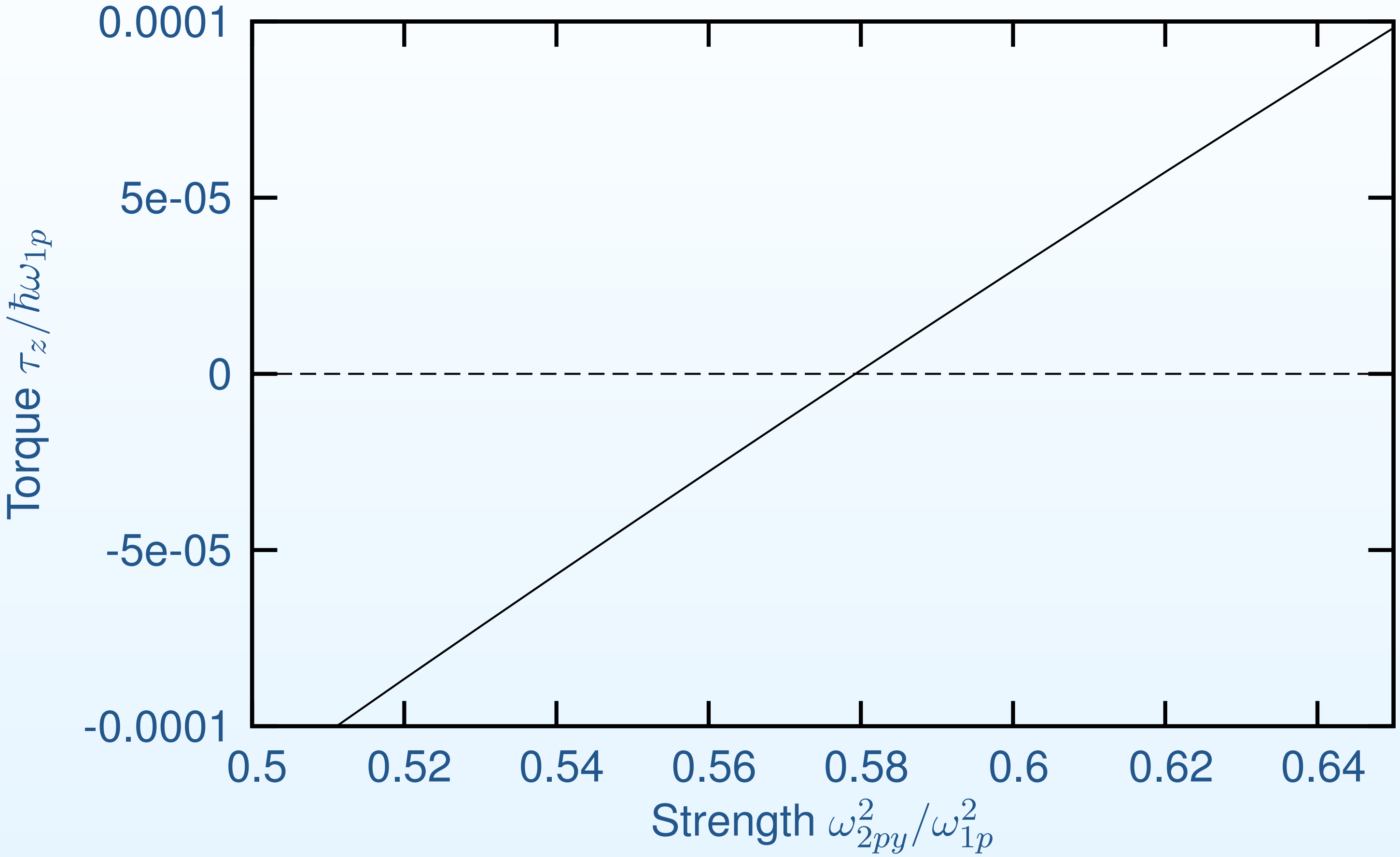
$$\omega_x = \omega_p,\ \omega_{1y} = 2\omega_{2y},\ \alpha = \pi/4,\ L = 0$$

Sign vs. L



$\omega_{1x} = \omega_{2x} = \omega_p$, $\omega_{1y} = \sqrt{2}\omega_p$, $\omega_{2y} = \omega_p/\sqrt{2}$, $\omega_{1px} = \omega_{1py} = \omega_{2px} = \omega_p$, $\alpha = \pi/4$.

Sign vs. strength



$\omega_{1x} = \omega_{2x} = \omega_{1p}$, $\omega_{1y} = \sqrt{2}\omega_{1p}$, $\omega_{2y} = \omega_p/\sqrt{2}$, $\omega_{1px} = \omega_{1py} = \omega_{2px} = \omega_{1p}$, $\alpha = \pi/4$, $L = 0.3c/\omega_{1p}$.

Conclusions

- Expression for torque between anisotropic surfaces in 1D, in terms of optical coefficients; they uncouple Casimir calculations from detailed models of materials.
- Results for anisotropic conductors, insulators, dicroic systems.
- Simple analytical formulae for ideal systems.
- $\tau \propto \hbar c/L$ (retarded), $\hbar\omega_p$ (nonretarded). A/L^2 correction expected in 3D.
- Sign may be modulated by changing L or by optically pumping dissimilar materials.
- Generalization to 3D.
- Experiments are on the way (Iannuzzi's talk).